THE
MATHEMATICAL GAZETTE.

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ON MODELS OF SPACE-FILLING SOLIDS.

THE most general form of a space-filling solid, in which there are no hollows or re-entering angles, is a tetrakaidekahedron (or fourteen-faced solid) the faces of which are eight hexagons,

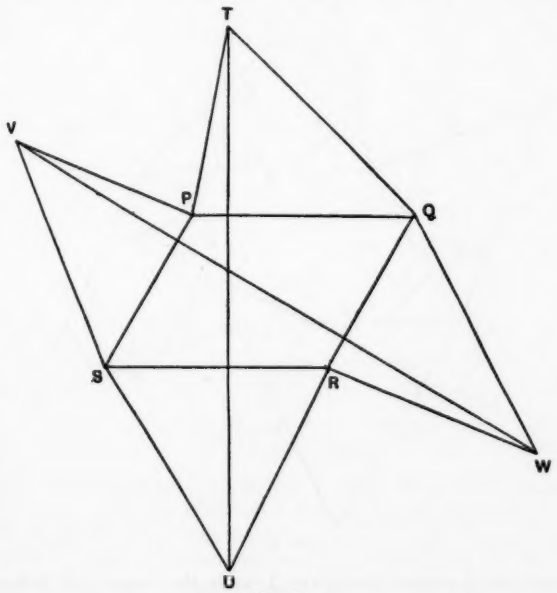


FIG. 1.

and six tetragons, with the condition that the opposite edges of each face are equal and parallel.

The fulfilment of this condition involves the division of the thirty-six edges of the solid into six groups of six edges each, the edges in each group being equal and parallel. It also involves the condition that the direction of the edges in each group should be the same as the direction of one pair of edges of some octohedron in which the opposite edges are parallel.

To make cardboard models of such solids the first step is to construct such an octohedron.

Draw any parallelogram $PQRS$ (figure 1) and draw TU perpendicular to the sides PQ and RS. Take any points T, U in the line TU, outside the parallelogram and on opposite sides of it, but so that of the three parts in which the line TU is divided by PQ and SR any two shall be greater than the third. Complete the triangles PTQ and RUS.

On PS draw the triangle PVS making $PV=PT$ and $SV=SU$, and on QR draw the triangle QWR making $QW=QT$ and $RW=RU$. To test the accuracy of the diagram join V and W, and ascertain that VW is perpendicular to PS and QR.

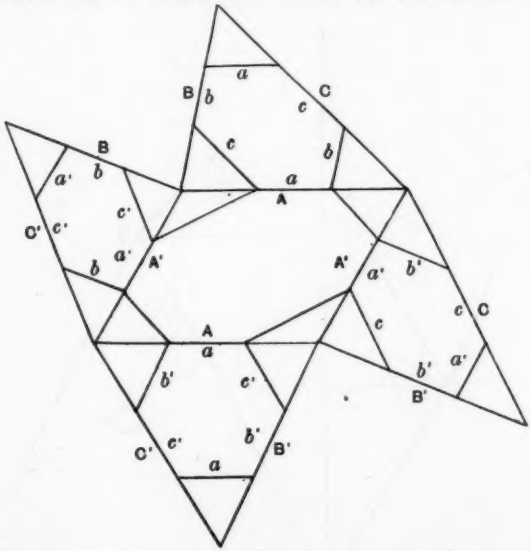


FIG. 2.

Figure 2 is the same as figure 1 with the lines and letters of construction omitted. Let A, A', B, B', C, C' be the lengths of the lines, and put these letters against the lines in the figure, whose lengths they represent.

Cut out a piece of cardboard the shape of figure 2 and cut half through the lines bounding the parallelogram. The triangles can then be folded so as to form a pyramid. Another pyramid can be constructed by cutting the line half through on the opposite side of the cardboard. The two pyramids joined base to base can form an octohedron in which the opposite edges are parallel. The edges may conveniently be referred to by their lengths A, A', etc.

From this octohedron a space-filling tetrakaidekahedron may be obtained by cutting off the corners by planes parallel to the diametrical planes in such a way as to leave each of the triangles in the form of a hexagon whose opposite sides are equal and parallel. The simplest way to do this is to cut off one third from each of the edges of the octohedron.

In figure 2 divide each of the sides of the triangles into three equal parts, and cut off the corners of the triangles and of the parallelogram as shown in the figure. Let a, a', b, b', c, c' be the lengths of the sides of the hexagons, a being derived from A, a' from A', and so on. Write these letters against the sides.

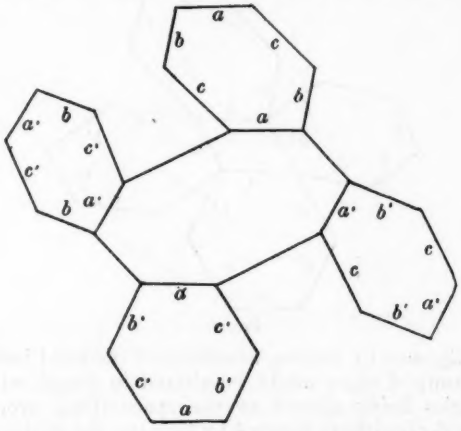


FIG. 3.

Figure 3 is the same as figure 2 with the corners removed. Cut out two pieces of cardboard the shape of figure 3, and cut half through the lines between the octagon and the hexagons, in one piece on the one side and on the other piece on the other side. Fold up the hexagons so as to join the pairs of edges marked b, b', c, c' respectively, and join the two structures octagon to octagon, taking care that the opposite edges are parallel. Six holes will be left in the form of parallelograms, two of which

have for their sides a and a', two b and b' and two c and c'. Close these holes with pieces of cardboard, and a space-filling tetrakaidekahedron is produced.

A more convenient mode of cutting out the cardboard is that shown in figure 4. Draw a parallelogram with the same angles as the original parallelogram, but having for its sides a and a'. Against the sides of this parallelogram place the same hexagons that are against the sides of the octagon in figure 3, but drawn as if they had been turned over. Cut out two pieces of cardboard the shape of figure 4, and cut half through the lines between the parallelogram and the hexagons, in one piece on one side, and in the other piece on the other side. Fold up the hexagons so as to bring together the pair of edges marked b, and also the pairs marked b', c, and c' respectively. Join the two structures by the edges a and a', taking care that the opposite edges come parallel to one another. There will remain four parallelograms having sides b and b', and c and c', respectively to be filled up. The result is the same tetrakaidekahedron which was obtained from figure 3.

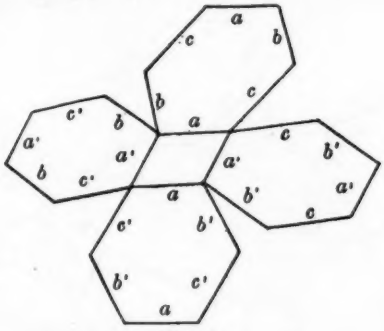


FIG. 4.

It is easily seen by looking at models of the solid just obtained that any group of edges might be altered in length without any of the angles being altered or the space-filling property lost. Figure 5 is derived from figure 4 by keeping the angles unaltered, but altering at random the lengths of the groups of edges. A solid may be constructed from figure 5 in the way described for constructing a solid from figure 4. This gives the most general form of space-filling solid without hollows or re-entering angles. The lengths a, a', β, β', γ, γ' have no relation to the lengths a, a', b, b', c, c', but the solid constructed from figure 5 can be placed in relation to the solid constructed from figure 4, and to the original octohedron, so that a, a, and A shall be parallel, and also a', a', and A'; and so on for the other edges.

As a group of edges may be of any length, let $\gamma' = 0$. Figure 5 is thus reduced to figure 6, and the resulting solid is changed to a

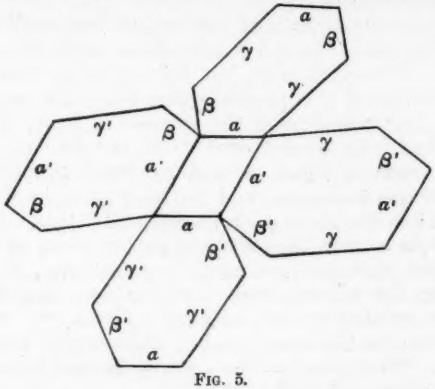


FIG. 5.

dodekahedron having for its faces four hexagons and eight tetragons.

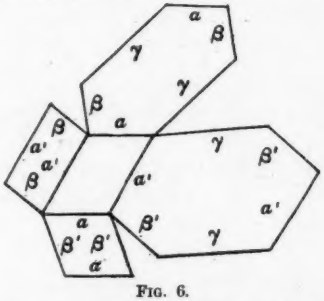


FIG. 6.

Put $\gamma' = 0$, and $\gamma = 0$, we get figure 7, which gives a dodekahedron whose faces are all tetragons.

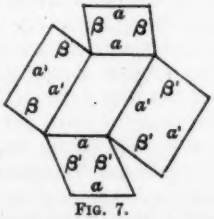


FIG. 7.

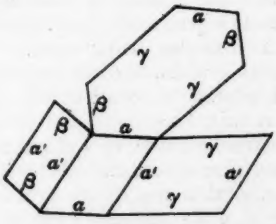


FIG. 8.

Put $\gamma' = 0$, and $\beta' = 0$, we get figure 8, which gives a six-sided prism bounded by plane ends.

Lastly put $\gamma'=0$, $\beta'=0$, $\gamma=0$, we get figure 9, which gives a parallelopiped.

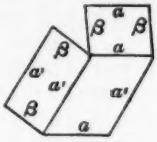


FIG. 9.

All these derived forms are space-filling solids.

[The paper was illustrated by cardboard models of all the solids described, in which the edges were coloured in six different colours, the edges A, a, and α having the same colour in all the solids, and so on for the other sets of edges. Of all the space-filling solids, except the last two, five solids were shown, in order to exhibit the way they could be packed together.]

It is easy to cut the most general space-filling solid out of a solid block. See figure 5—Cut a plane surface on which draw the parallelogram aa'. Through the sides cut four planes and along the new edges take the lengths β, β', γ, γ' respectively. Complete the hexagons. Through the edges β, β' cut planes, and also through the edges γ and γ'. Complete the parallelograms $\beta\beta'$ and $\gamma\gamma'$. We have now three sides of the other four hexagons. Complete these hexagons. Cut the remaining plane through a and a', and the solid is finished. WALTER BAILY, M.A.

A METHOD OF CALCULATING APPROXIMATELY HIGH POWERS AND ROOTS OF NUMBERS, *WITHOUT THE AID OF TABLES.*

I. POWERS.

THE nth power of a number a, when n is a whole number, may be defined as "the final result when we multiply 1 by a, then multiply the resulting product by a, then that result by a, and so on till n multiplications have been performed." But when n is itself a power of 2, *e.g.* when $n=8$, the number of multiplications may be diminished. Given a, we can get a^8 by 3 multiplications, viz., by multiplying a by a which gives a^2, then multiplying a^2 by a^2 which gives a^4, then a^4 by a^4 which gives the required a^8.

Again $a^{11}=a^{8+2+1}=a^8\times a^2\times a$. Here a^2 and a^8 can be got by 3 multiplications, and a^{11} by multiplying a^8 by a^2, and the result by a, so that we get a^{11} by a total of 5 multiplications instead of 10. But we can also arrange the operations as indicated by the formula $a^{11}=\{(a\times a)^2\times a\}^2\times a$, which requires also 5 multiplications to be performed. In what follows we shall use the latter arrangement. Either way, the economy in actual figuring is not

great if an *exact* result is wanted, but if only an approximation is required, to a certain number of significant figures, the economy may be considerable, and the higher the power, the greater will be the economy.

Thus, to find the 11th power of 2·1251 to 5 significant figures, the actual figuring may be carried out as follows:

2·1251	$=a$
4·2502	
2125	
425	
106	
2	
4·5160	$=a^2$
18·0640	
2 2580	
452	
271	
20·3943	$=a^4$
2·1251	
40·7886	
2 0394	
4079	
1020	
20	
43·340	$=a^{4+1}$
1733·60	
130 02	
13 00	
1 73	
1878·35	$=a^{5+2}$
2·125 1	
3756·70	
187 84	
37 57	
9 39	
18	
3991·68	$=a^{8+2+1}=a^{11}$

Answer correct to 5 figures, 3991·7.

This method may be systematized as follows:

Express the index n as a number in the scale of 2.

Thus 11 (scale of 10) = 1011 (scale of 2).

By squaring a we get the power whose index is 10 (scale of 2).

" " a^2 " " " 100 "

" multiplying by a " " " 101 "

" squaring again " " " 1010 "

" multiplying by a " " " 1011 "

Thus by a series of operations of squaring and multiplying by a we can get any power of a that we may want. The rule will now be sufficiently obvious without stating it in general terms.

II. Roots.

A similar method will apply to the calculation of *roots*, enabling us by a combination of operations of *extracting square-roots* and *multiplying by b* to find to any degree of approximation *any root*, or (more generally) any fractional power of a number b.

Thus, to find the 11th root of 3991·7 ($=b$), let us express 1/11 as a radical fraction in the scale of 2. This may be at once written down. It is $\cdot\dot{0}001011101\dot{0}\dot{1}$. Now, since $2^{10}=1024$, it is clear that in order to have the same degree of approximation in the scale of 2 as in the scale of 10 we must have about 10/3 times the number of significant figures, so that $\frac{1}{11}$th to the degree of approximation we have been working to, would have to be taken as ·0001011101000101110 1, which would require the performance of 20 square-root extractions, and 9 multiplications by b, each carried to 5 significant figures on the scale of 10. But we may economize by using this method to find a rougher approximation to the root required, and then use another method to get a closer approximation.

Take $\frac{1}{11}\doteq\cdot00011000$ in the scale of 2, which will be true to about 5 significant figures, *i.e.* to less than $\frac{1}{2}$ per cent.

Performing the calculations, then, to 3 significant figures in the scale of 10, we may take $b\doteq4000$, and carry out the actual figuring as follows:

6 | 4000	123 | 36	$63\cdot2=b^{\cdot1}$ (index in scale of 2)
	400	
126	369	
	31	

$$63\cdot2\times4000=252800=b^{1\cdot1}$$

5 | 252800	105 | 25	$505=b^{\cdot11}$
	2800	
2 | 505	42 | 4	$22\cdot5=b^{\cdot011}$
	105	
44	84	
	21	
4 | 22·5	87 | 16	$4\cdot73=b^{\cdot0011}$
	6·5	
	6·2	
	3	
2 | 4·73	41 | 4	$2\cdot18=b^{\cdot00011}=b^{\frac{3}{32}}$
	·73	
	41	
	32	

Thus the 11th root of 3991·7 is found to lie between 2·1 and 2·2.

To find a closer approximation we may proceed thus. Remembering that ·00011 is in excess of 1/11 by 1/352, we may take it that 2·1 is probably a better approximation than 2·2. Calculate, then, $(2\cdot1)^{1011}$ (the index being in the scale of 2).

$$\begin{array}{l} 2\cdot1 \\ \hline 4\cdot2 \\ \quad 21 \\ \hline 4\cdot41 \\ \hline 17\cdot64 \\ \quad 1\,76 \\ \quad\quad 4 \\ \hline 19\cdot94 \\ \hline 38\cdot88 \\ \quad 1\,94 \\ \hline 40\cdot82 \\ \hline 1632\cdot8 \\ \quad 32\,7 \\ \quad\quad 8 \\ \hline 1666\cdot5 \\ \hline 3333\cdot \\ \quad 167 \\ \hline 3500\cdot \end{array} \begin{array}{l} = c \\ \\ \\ = c^{10} \\ \\ \\ = c^{100} \\ \\ \\ = c^{101} \\ \\ \\ = c^{1010} \\ \\ \\ = c^{1011} = (2\cdot1)^{11}. \end{array}$$

Now let $(2\cdot1+x)^{11} = 3991\cdot7$;

$$\therefore \left(1+\frac{x}{2\cdot1}\right)^{11} \doteqdot \frac{3991\cdot7}{3500} = 1+\frac{491\cdot7}{3500};$$

$$\therefore \frac{11x}{2\cdot1}+\frac{11\cdot10}{1\cdot2}\left(\frac{x}{2\cdot1}\right)^2+\frac{11\cdot10\cdot9}{1\cdot2\cdot3}\left(\frac{x}{2\cdot1}\right)^3 \doteqdot \frac{491\cdot7}{3500};$$

$$\therefore \frac{11x}{2\cdot1} \doteqdot \cdot14049-\frac{5}{11}\left(\frac{11x}{2\cdot1}\right)^2-\frac{5}{11}\cdot\frac{3}{11}\left(\frac{11x}{2\cdot1}\right)^3$$

$$\doteqdot \cdot1405-\frac{5}{11}(\cdot1405)^2-\frac{5}{11}\cdot\frac{3}{11}(\cdot1405)^3$$

$$\doteqdot \cdot1405-\cdot0093-\cdot0004$$

$$\doteqdot \cdot1308;$$

$$\therefore x \doteqdot \cdot02497.$$

Thus $\sqrt[11]{3991\cdot7} \doteqdot 2\cdot1+\cdot02497$

$$\doteqdot 2\cdot125.$$

To save space, some of the actual figuring has been omitted in writing out this second approximation.

I omit also the discussion of the degree of exactness that ought to be expected. A third approximation might be made on the same lines as the second, putting $(2\cdot125+y)^{11}=3991\cdot7$; and indeed, once an approximation to within 1% has been made, there are many ways of getting to any required degree of approximation. Another method for the second and higher

approximations would be, after calculating the 11th power of 2·1, to calculate also that of 2·2, which is nearly 5843·2. Then, using "proportional parts," we should have for second approximation

$$2{\cdot}1 + {\cdot}1 \times \frac{3991{\cdot}7 - 3500}{5843{\cdot}2 - 3500},$$

which $\doteqdot 2{\cdot}121$, and is true only to the 3rd figure.

To calculate approximately such a quantity as $(3{\cdot}14)^{{\cdot}243}$, we should first express ·314 as a radix-fraction in the scale of 2, viz. ·00111110001101.... If we take ·0011111 as the approximate value of this (the error being less than 1 %), then by 7 square-root extractions and 4 multiplications, carried to 3 figures, we shall get the result correct to about 1 %. R. F. MUIRHEAD.

THE STUDY OF MATHEMATICS.

MR. BERTRAND RUSSELL'S recent article* on the study of mathematics is of very great importance to our ideas and ideals of teaching mathematics, and should lead to some fruitful discussion. As one might expect from such a keen and subtle logician,—who did not neglect, by the way, to point out, in this *Gazette*, some of the absurd errors in Euclid which, to his mind, diminish its educational value,†—he objects (p. 32) to the men who maintain that mathematics trains the reasoning faculties, and yet are: "unwilling to abandon the teaching of definite fallacies. And the reasoning faculty itself is generally conceived, by those who urge its cultivation, as merely a means for the avoidance of pitfalls, and a help in the discovery of rules for the guidance of practical life. All these are undeniably important achievements to the credit of mathematics; yet it is none of these that entitle mathematics to a place in every liberal education." The ordered cosmos of mathematics has a high and noble aesthetic value: "mathematics endeavours to present whatever is most general in its purity, without irrelevant trappings" (p. 34).

And yet it is no highly abstract and seemingly impracticable form of introduction to mathematics that Mr. Russell recommends; it is by a practical method that geometry is to be taught, and his remarks (pp. 34-37) on this subject, and on the teaching of arithmetic, algebra, and the calculus are deeply interesting as giving Mr. Russell's ideas as to "How should the teaching of mathematics be conducted so as to communicate to the learner as much as possible of this high ideal?"

À propos of the calculus, Mr. Russell remarks (p. 37) that modern researches (into the theory of aggregates and into logic), to all candid and inquiring minds, have facilitated the mastery of higher mathematics, which hitherto has not been treated exactly. In this, it seems to me, is suggested a debateable point. It is incontestably true that modern analysis appeals to a logician while the infinitesimal calculus may appeal in vain. But, because of the apparently *natural* growth of infinitesimal methods in so many minds,‡ and the use, to this day, of those methods in geometry and physics, in preference to a rival which is, at least, *expressed* in a way more in conformity with logic, demands an inquiry.

* "The Study of Mathematics." By the Hon. Bertrand Russell; *The New Quarterly; a Review of Science and Literature*, No. 1, Nov. 1907, pp. 31-44.

† Cf. also *The Principles of Mathematics*, vol. i., Cambridge, 1903, pp. 404-407.

‡ Even Newton (as De Morgan showed) had *at first* an infinitesimal method.

In certain stages of human development, errors have played a great part in the exact sciences. Infinitesimal methods contained errors which all mathematicians have felt, explained either with partial correctness, wrongly, or not at all, and then overlooked for the sake of the great truths contained in this method, which, strange to say, seem only to be able to exercise their full power when they are manipulated by a logically-faulty calculus. These faults have been almost immortalised by the convenient symbols d and $\int$.

There is another kind of error, of which the best example seems to be the formalist theories of numbers, in which a 'number' is said to be a mere *sign*—not a sign *for* something, but an empty symbol with which we are to 'calculate' according to certain rules. This error has been committed by very many mathematicians of great eminence, and yet assuredly any child would instinctively reject such an absurd view. Subtle modern logicians like Frege and Russell share the child's point of view, except that they are able to give reasons for their instinct. Here seems to be an interesting problem of psychology; my historical work has forced me to hold the view that an advanced knowledge of mathematics, without a correspondingly advanced knowledge of logic, renders people more liable to absurdities in views than comparative ignorance does, and that it is only when logic has developed in the mind (as it must, even with mere mathematicians, when they come to consider the principles of mathematics) that views resembling those of an intelligent child, but no longer held for "intuitive"—unanalysed —reasons, are attained. Here one would be glad to have practical teachers' opinions; the whole question concerns teaching very nearly.

To return to Mr. Russell, he lays stress (pp. 37-40) on the importance of inquiry into principles as a part of education, and emphasises (p. 40) the fact, which most mathematicians persist in overlooking, that "... the true principles are as much a part of mathematics as any of their consequences."

Then, too, Mr. Russell remarks (p. 41) that the objects considered in mathematics have, in the past, been suggested by phenomena,* yet mathematics, on the one hand, and ourselves, our thoughts, and the whole universe of existing things, on the other, are mutually independent. This is the point of view to which the work of Stallo, Mach, Kirchhoff, Hertz, Pearson, and others in the theory of knowledge, leads; I mean the distinction of nature and the purely logical 'images' of it with which we deal in science.

Finally, Mr. Russell deserves the thanks of all for the clothing of truth in striking and paradoxical garments; he tells us that the making of machines, the travelling from place to place, and the victory over foreign nations, whether in war or commerce, are all "of doubtful value" (p. 32), and (p. 44): "indirectly the mathematician often does more for human happiness than any of his more practically active contemporaries."

Jan. 1st, 1908. PHILIP E. B. JOURDAIN.

REVIEWS.

A First Course in the Differential and Integral Calculus. By W. F. OSGOOD. Pp. xv, 423. New York: The Macmillan Company. 1907.

A Treatise on the Integral Calculus, founded on the Method of Rates. By W. WOOLSEY JOHNSON. Pp. v, 440. New York: John Wiley & Sons. 1907.

No one who wishes to realise the enormous advance that has been made during the last ten or twenty years in the teaching of the Calculus could do better than to read and contrast these two books, published in the same country

* Consider the development of the idea of a *function* in mathematics to the fundamental idea in the logic of relations (Dedekind, Peano, Schröder, Russell, and others).

and in the same year, but differing completely in their objects and methods and representing one the old school and the other the new.

It is no disparagement of Prof. Johnson's book to say that it might have been written thirty years ago. It is a book of a frankly old-fashioned type, another *Todhunter* or *Williamson*. It may be better or worse than *Todhunter*; on the whole we prefer it to *Todhunter*: but it is with such books as *Todhunter* that we must compare it.

We must not be understood as implying that Prof. Johnson's book may not be of considerable utility. There is plenty of room for an improved *Todhunter*. It is, as Prof. Osgood himself remarks, of the utmost importance that the formal side of the Calculus should be taught thoroughly in a first course; no one can be a good mathematician who cannot differentiate

$$\arccos\left(\frac{a\cos x+b}{a+b\cos x}\right),$$

or show a little ingenuity in devices of integration. And it is at anyrate open to argument that the best way to impart this most necessary training is by means of a course in which the theoretical difficulties clustering around the foundations of the Calculus are deliberately or unconsciously ignored. That some of these difficulties should, at first, be passed over in silence, or almost in silence, is indeed obviously necessary. It would be ridiculous to insist that nobody should learn how to determine maxima and minima until he can prove that a continuous function attains its upper and lower limits, or to calculate areas until he can prove their existence. But there are different ways of passing over difficulties. We may simply and absolutely ignore them: that is a course for which there is often much to be said. We may point them out and avowedly pass them by; or we may expand a little about them and endeavour to make our conclusions plausible without professing to make our reasoning exact. There is only one course for which no good defence can ever be found. This course is to give what profess to be proofs and are not proofs, reasoning which is ostensibly exact, but which really misses all the essential difficulties of the problem. This was Todhunter's method, and it is one which Prof. Johnson too often adopts.

Thus, when he professes to prove that "an integral can be differentiated with respect to a quantity independent of the current variable and the limits by differentiating the expression under the integral sign," he is professing to prove what he himself later on acknowledges to be untrue by arguments entirely destitute of validity. And what is gained? Why not simply say, "it is natural to suppose, and can be proved to be in fact often true, that an integral . . . "? The reader would find the applications of the principle just as interesting, and the author, when confronted with the necessity of explaining why what he says he has proved is not true of the integral

$$\int_0^\infty \frac{\sin ax}{x}\,dx,$$

would not have been obliged to invent the untrue and absurd explanation that "this results from the fact that, although a occurs in it, the expression is not really a function of a."

But let us attempt to judge the book by more appropriate standards. The first two chapters, on methods of integration, seem to us fairly good and interesting, though far from ideal. There is a lack of system about the treatment. Surely, in 125 pages, more might be said as to what are the general classes of functions whose integrals can always be found. It is not even pointed out that

$$\int R(\cos x,\ \sin x)dx$$

can always be evaluated by the substitution $\tan\frac{1}{2}x=t$, although this substitution is used in particular cases; and the treatment of algebraical functions is equally unsystematic.

The most interesting sections are those on approximate integration (in Ch. III.) and on applications to probability (in Ch. IV.). The chapter on definite integrals (Ch. V.) is deplorable theoretically, but contains many interesting results and examples. But we could wish that Ch. VI. had been omitted. The end of a

text-book on the Integral Calculus is not the right place for an introduction to the theory of functions of a complex variable, and the introduction itself is neither sound nor particularly clear.

Prof. Osgood's book is, of course, a work of an entirely different character. His task is a much more difficult one, for he is showing the way over a road the roughness of which he knows. On the whole, his book seems to us the best elementary treatise on the Calculus in the English language; only *Lamb* and *Gibson* seriously compete with it, and of the three books the latest seems to us the most lucid, the best arranged, and (what is very important) the most *level* in its standard of interest and difficulty. Unfortunately there is hardly enough technical and analytical complication to suit a Cambridge course. If there is one chapter which we would single out for praise it is Ch. XIV., on "partial differentiation," "the total differential," "small errors," and so on; or perhaps Ch. X. on "Mechanics," the 30 odd pages of which might well supersede the whole of some well-known text-books of "Dynamics."

The book is one, we fear, likely to make considerable demands on the knowledge and discrimination of the teacher; for Prof. Osgood is a firm believer in a *gradual* introduction to fundamental difficulties, and anyone using the book as a text-book must be ready to parry the occasional awkward question which comes a little before its time. For this reason we sometimes wish for more detail in the text. It is a big jump from the rather innocent attitude towards limits adopted at the beginning to the formal definition for functions of two variables given in Ch. XIV. In the case of one variable no such definition is ever given. And we should have liked to have seen included at anyrate a short and popular sketch of Dedekind's theory of number. It is not very difficult to explain in general terms, and enables a good many gaps to be filled in, notably in connection with the 'Fundamental Principle' (p. 246) that an increasing function approaches a limit or tends to infinity.

One minor point suggests itself. We do not like the notation "$\lim_{x=a}$." Messrs. Leathem and Bromwich's "$x \to a$" is much clearer and much less ugly. And why "$\lim_{n=\infty}$"? It seems to us even more essential to use a different symbol here. For Prof. Osgood does not need to be told that nothing is ever "equal to ∞."

G. H. HARDY.

MATHEMATICAL INSTRUMENTS.

DR. ERSKINE MURRAY'S **Patent Anglemeter.** (Messrs. W. J. George, Ltd., Great Charles Street, Birmingham.)

This is described as a modified sextant in which one of the mirrors is omitted and the telescope replaced by a pinhole sight. It is intended for measuring angles in any plane. When mounted it has the advantage over a theodolite in that the angle is completely measured by a single reading. It is intended mainly for use in schools, and is issued at the low price of 2s. 9d, or of superior quality, with vernier, at 8s. 6d. It will be found a valuable addition to the school equipment, and under certain circumstances will be found of great use in ordinary surveying. Indeed, its value in this respect has been amply demonstrated, as the preliminary survey of the Krugersdorp to Mafeking Railway was made by its means. All difficulties arising from inequality of surface may be avoided, and we are told that surveys can be made as accurately as with a chain. It is recommended by Messrs. Simmons & Richardson in their *Practical Geography*, and its value to the teacher is endorsed by Profs. Greenhill, Perry and Robinson. It can be used to find the distance of an inaccessible object, or the width of a river, to draw sketch maps, for working out many of Euclid's propositions out-of-doors with the aid of a tape measure, for measuring the angle of dip, for contouring ground which is not level, as an optical square for offsets, and for range finding. In a word, it is a type of instrument which has come to stay.

A Tangentometer is supplied by Messrs. Harris of Birmingham at the price of 13s. 6d., after the design of Mr. D. Berridge. It consists of a boxwood metre rule; a semi-circle graduated in degrees; a half-metre boxwood rule perforated at various points, and with a slot to read degrees on the semi-circle; a rule graduated in mms. with a slot; and a plumb bob and cursor. We can imagine

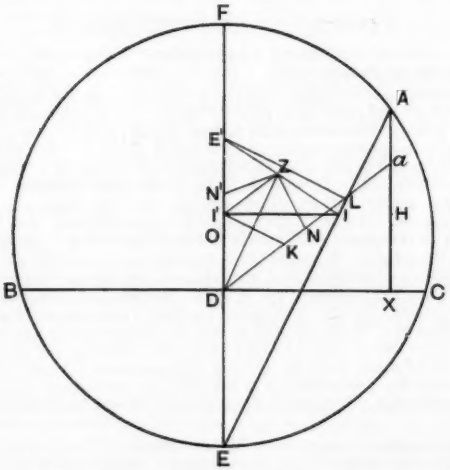
no better way of introducing the beginner to the trigonometrical ratios than by making him calculate them in the laboratory with such a contrivance as this. It is claimed that results may be obtained after very little practice correct to within a half per cent. To those who have to teach what is called "practical mathematics" the instrument will come as a boon and a blessing. It is strongly put together, and we should think that children may be introduced to the trigonometrical ratios at a much earlier age than is wise or even possible at present without such assistance.

Slide-Rules. From Messrs. Griffin & Sons, Kingsway, London, we have received two slide-rules which will be welcomed in schools where financial considerations are important; and where, indeed, are they not? They are of two sizes, 5 in. and 10 in., and cost respectively 1s. and 2s. each. They are not intended to take the place of the more expensive instruments, but will be found extremely useful to accustom the beginner to the value of the rule as a weapon in the armoury of the calculator. The rule is specified as made of 3-ply card on which the divisions are printed with a degree of accuracy and clearness which is not far behind that of those printed divided on celluloid. The claim is entirely justified, and the makers may be congratulated on the foresight which has enabled them to bring into the market instruments for which there should be a wide demand. Our only fear is that the edges of the card may too readily split under the nervous picking to which so many children are subject. It would be well that, at any rate at first, the manipulation of the instrument should take place under the direct guidance of the master.

MATHEMATICAL NOTES.

264. [K. 2. d.] *Two Geometrical Proofs of Feuerbach's Theorem.*

I. Employing the usual notation; along DF take $DE'=ED$ and $DN'=\frac{1}{2}R=DN$. Bisect IE' in Z, and draw II' perpendicular to DF. Join ZD, ZN, ZN', ZI'. Since DZ is parallel to EI and $=\frac{1}{2}EI$, and $EI^2=2R.ED$,



therefore $DZ^2=DN'.DE'$ and angle $DZN=E'$. But DZ bisects NDN' (as DN is parallel to OA); therefore the triangles DZN, DZN' are congruent

and $ZN=ZN'$. Therefore angle $NZN'=$twice $DZN'=2E'$. From the right-angled triangle $II'E'$, angle $IZI'=2E'$. Thus angle $NZN'=IZI'$ and therefore angle $IZN=I'ZN'$. The triangles IZN, $I'ZN'$ are therefore congruent and

$$NI=N'I'=DN'-DI'=\tfrac{1}{2}R-r.$$

Cuddapah, India. V. RAMASWAMI AITAR.

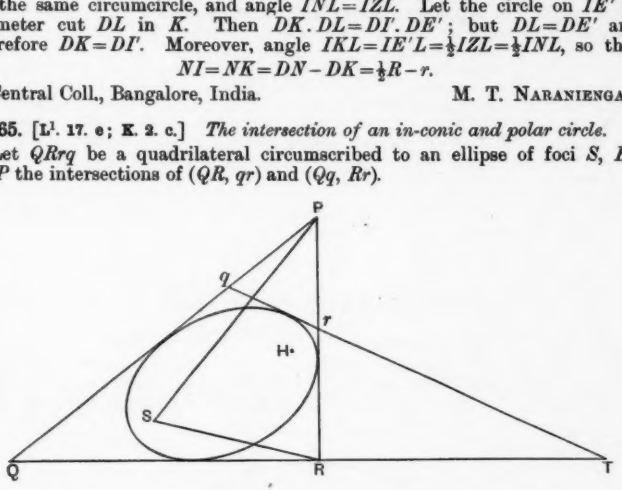
II. Let DN meet AE in L; so that $DL=ED=DE'$ and $E'L$ is perpendicular to AE. ZIL is an isosceles triangle and DZ being parallel to IL touches at Z the circumcircle of ZIL. But $DZ^2=DN.DL$, therefore N lies on the same circumcircle, and angle $INL=IZL$. Let the circle on IE' as diameter cut DL in K. Then $DK.DL=DI'.DE'$; but $DL=DE'$ and therefore $DK=DI'$. Moreover, angle $IKL=IE'L=\frac{1}{2}IZL=\frac{1}{2}INL$, so that

$$NI=NK=DN-DK=\tfrac{1}{2}R-r.$$

Central Coll., Bangalore, India. M. T. NARANIENGAR.

265. [L¹. 17. e; K. 2. c.] *The intersection of an in-conic and polar circle.*

Let $QRrq$ be a quadrilateral circumscribed to an ellipse of foci S, H; T, P the intersections of (QR, qr) and (Qq, Rr).



Then if the tangents from Q, R, r, q subtend respectively angles α, β, γ, δ at S

$$PSq=\tfrac{1}{2}(2\gamma+2\delta)-\delta=\gamma,$$

$$TSR=\tfrac{1}{2}(2\beta+2\gamma)-\beta=\gamma,$$

or $PSq=TSR$. When qr is parallel to QR, T goes to infinity and $PSq=SRQ=HRP$. The triangles PqS, PHR are therefore in this case similar, and $Pq.PR\,(=Pr.PQ)=SP.HP$.

This theorem is due to Mr. E. P. Rouse; and is to be found in an article on "The Director Circle of an Inscribed Conic" in No. 2 of the *Mathematical Gazette*, July, 1894.

Mr. Rouse's application of this property to prove that the director circle of a conic inscribed in a triangle intersects orthogonally the polar circle of the triangle is here given for the benefit of a younger generation:

Draw RM (perpendicular to PQ) to meet PD (perpendicular to QR) in E the orthocentre of PQR.

From the well-known formula for the cosine of the angle between the tangents from an external point P (as given in Salmon and elsewhere)

$$\begin{aligned}2SP.HP\cos QPR&=SP^2+HP^2-4CA^2\\&=2(CP^2+CS^2)-4CA^2\\&=2\{CP^2-(CA^2+CB^2)\}.\end{aligned}$$

Therefore $CA^2+CB^2=CP^2-SP.HP\cos QPR$

$=CP^2+Pq.PM$, since $Pq.PR=SP.HP$

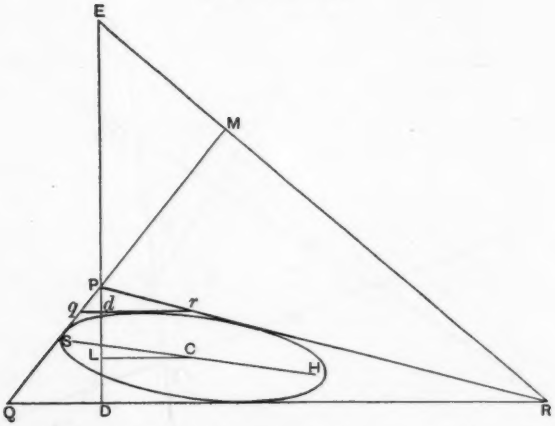
$=CP^2+Pd.PE$, since $EMdq$ is cyclic

$=CE^2-(CE^2-CP^2)+Pd.PE$

$=CE^2-(EL^2-PL^2)+Pd.PE$

$=CE^2-EP(EL+PL-Pd)$

$=CE^2-EP.ED$,



for CL lies halfway between the parallels QR, qr.

Thus $CE^2=CA^2+CB^2+EP.ED$. R. F. Davis.

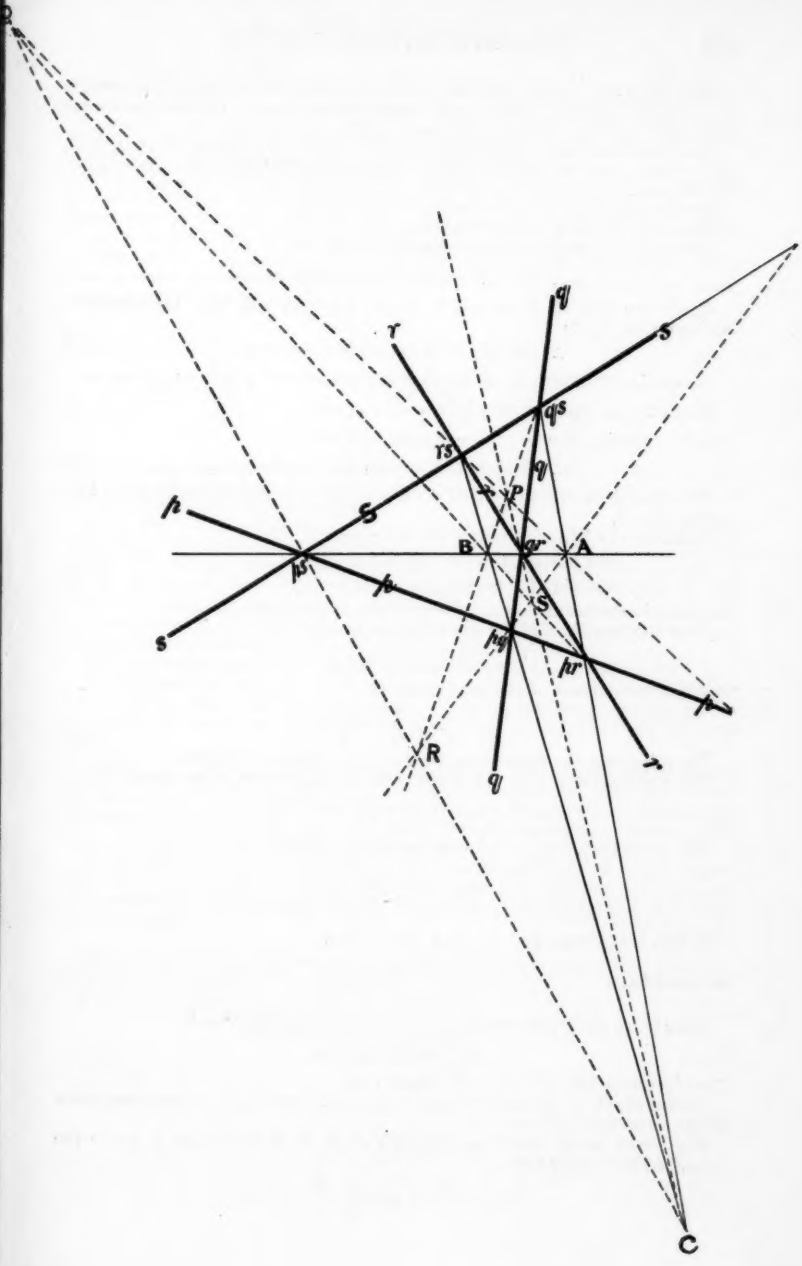
Mr. Rouse adds:—"If we are willing to extend the term 'polar circle' to the case of an acute-angled triangle, a similar geometrical proof shews that the director circle cuts this polar circle at the extremities of a diameter of the latter."

266. [K. 1. c.]. *The Self-Reciprocal* 13*-gon.*

A triangle considered as 3 points joined by 3 lines corresponds reciprocally to a triangle considered as 3 lines intersecting in 3 points. It may thus be said to be a self-reciprocal figure.

A quadrangle consisting of 4 points joined by 6 lines corresponds reciprocally to a quadrilateral consisting of 4 lines intersecting in 6 points. Neither of these figures therefore is self-reciprocal, but by a combination of them the self-reciprocal figure which it is the object of this note to explain can be obtained.

Take four lines p, q, r, s intersecting in 6 points (pq), etc., and draw its diagonal $\triangle ABC$ (so that AB passes through (ps) and (qr)). Then A, B, C can each be joined to 2 of the 6 points, viz. A to (rs) and (pq), etc., thus giving 6 lines. If these 6 lines are drawn (and a different coloured pencil might conveniently be used to distinguish them from the previous lines) they will be found to intersect by threes in four points, say P, Q, R, S, and when these four points have been marked the self-reciprocal 13-gon has been constructed.


Q
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C

The figure can equally well be obtained starting from P, Q, R, S; and it may be described as consisting of a quadrilateral p, q, r, s and a quadrangle P, Q, R, S having a common diagonal $\triangle ABC$.

The proof that the 6 lines, stated above to concur by threes at P, Q, R, S, really do so, can be obtained using trilinear coordinates, with $\triangle ABC$ as $\triangle$ of reference.

Let the line s be $l\alpha+m\beta+n\gamma=0$.

Then the join of A to (rs) is $m\beta+n\gamma=0$.

Now (rs), (pq) are harmonic conjugates to B, C;

$\therefore$ the join of A to (pq) is $m\beta-n\gamma=0$. .....(i)

Again the join of B to (qs) is $l\alpha+n\gamma=0$, and (qs), (pr) are harmonic conjugates to A, C;

$\therefore$ the join of B to (pr) is $l\alpha-n\gamma=0$. .....(ii)

Comparing this with (i) we see that the equation of p is $l\alpha+m\beta-n\gamma=0$.

Similarly the equation of q is $l\alpha-m\beta+n\gamma=0$,

and the equation of r is $-l\alpha+m\beta+n\gamma=0$;

$\therefore$ the join of C to (qr) is $l\alpha-m\beta=0$. .....(iii)

Now the three lines (i) (ii) (iii) intersect at the point $l\alpha=m\beta=n\gamma$, which is the point S.

Similarly the join of A to (rs) viz. $m\beta+n\gamma=0$,

the join of B to (qs) viz. $l\alpha+n\gamma=0$,

and the join of C to (qr) viz. $l\alpha-m\beta=0$

intersect in the point $l\alpha=m\beta=-n\gamma$ which is the point P.

Thus the four lines of the quadrilateral are

$$l\alpha\pm m\beta\pm n\gamma=0,$$

and the four corners of the quadrangle are

$$\frac{1}{l},\quad \pm\frac{1}{m},\quad \pm\frac{1}{n}.$$

To reciprocate the figure into itself we can proceed as follows:

The equation of a conic for which ABC is self-polar is of the form

$$u\alpha^2+v\beta^2+w\gamma^2=0.$$

The polar of $\frac{1}{l},\ \pm\frac{1}{m},\ \pm\frac{1}{n}$ with respect to this is

$$\frac{u\alpha}{l}\pm\frac{v\beta}{m}\pm\frac{w\gamma}{n}=0.$$

If this is the same as $l\alpha\pm m\beta\pm n\gamma=0$,

we must have $$\frac{u}{l^2}=\frac{v}{m^2}=\frac{w}{n^2}.$$

Thus if we reciprocate with respect to the (imaginary) conic

$$l^2\alpha^2+m^2\beta^2+n^2\gamma^2=0,$$

A reciprocates into BC etc., and S into s etc.

Conics which either pass through P, Q, R, S or touch p, q, r, s are necessarily of the form $u\alpha^2+v\beta^2+w\gamma^2=0$.

For conics which both pass through P, Q, R, S and touch p, q, r, s the equations for u, v, w are

$$\frac{u}{l^2}+\frac{v}{m^2}+\frac{w}{n^2}=0 \text{ and } \frac{l^2}{u}+\frac{m^2}{v}+\frac{n^2}{w}=0,$$

thus the only such conics are of the form

$$l^2\alpha^2+m^2\omega\beta^2+n^2\omega^2\gamma^2=0$$

and

$$l^2\alpha^2+m^2\omega^2\beta^2+n^2\omega\gamma^2=0,$$

when ω and ω^2 are the imaginary cube-roots of unity.

The 13-gon is not a closed figure in the sense that every pair of points is joined: I believe the $\triangle$ is the only figure of this kind possible, though a figure such as a square with its diagonals drawn may appear to be so owing to some of the points of intersection of the sides having moved to infinity.

Each of the six lines such as PS intersects two of the original lines viz. p and s at fresh points: These points are 12 in number, and their coordinates will be found to be

$$\frac{2}{l},\ \pm\frac{1}{m},\ \pm\frac{1}{n},$$

$$\pm\frac{1}{l},\ \frac{2}{m},\ \pm\frac{1}{n},$$

and

$$\pm\frac{1}{l},\ \pm\frac{1}{m},\ \frac{2}{n}.$$

If anyone has the energy, as one of my pupils Mr. D. F. Ferguson has had, to work out the equations of the 66 joins of these points he will be rewarded by some curious results.

18 of the 66 joins coincide with one or other of the lines already found: 12 more, of the forms $l\alpha\pm2m\beta=0$, $l\alpha\pm2n\gamma=0$ etc., pass through one of the points A, B, C: 12 more, of the forms $3l\alpha\pm m\beta\pm n\gamma=0$, pass through one of the 6 corners of the original quadrilateral. The remaining 24 are of the form $\lambda l\alpha\pm\mu m\beta\pm n\gamma=0$, when λ, μ, ν stand for 1, 3, 5 in any order.

These do not pass through any of the original 13 points, but a large number of concurrences on the original lines can be detected.

One particular case of the 13-gon is obtained by taking $l=m=n=1$.

In this case the corners of the quadrangle are the inscribed and escribed centres of $\triangle ABC$, and the six corners of the quadrilateral are the points where the internal and external bisectors of the angles meet the opposite sides.

Again if l, m, n are taken proportional to a, b, c, the sides of the quadrilateral become the line at infinity and the joins of the mid-points of the sides of $\triangle ABC$: while the sides of the quadrangle are the 3 medians, and the 3 parallels to the opposite sides through the corners: these 6 lines determine four points of intersection, one being the C.G. of the $\triangle$.

If any member of the Association finds his pupils inclined to get mixed between a quadrangle and a quadrilateral, may I recommend him to make them draw the 13-gon beginning first from the quadrilateral and then from the quadrangle.

C. O. TUCKEY.

267. [V. a.] Supposing a^m to be defined by the property $a^m\times a^n=a^{m+n}$, to prove $(a^m)^n=a^{mn}$.

From the definition we have

$$\text{if } \phi(x)\times\phi(y)=\phi(x+y).$$

Then $$\phi(x)=z^x,\ \phi(y)=z^y.$$

Now let $$a^{mx}=f(x).$$

Then
$$\begin{aligned} f(x).f(y)&=a^{mx}.a^{my}\\ &=a^{mx+my}\\ &=a^{m(x+y)} \quad \{\text{Law of multiplication}\}\\ &=f(x+y);\end{aligned}$$

$$\therefore\ f(x)=z^x;$$

$$\therefore\ f(1)=z,$$

but
$$f(1)=a^m;$$
$$\therefore\ z=a^m;$$
$$\therefore\ a^{mx}=(a^m)^x.$$

(No assumption is made as to the values of the quantities a, m, n.)

Cor. Demoivre's Theorem.

Let $$f(\theta)=\cos\theta+i\sin\theta.$$

Then
$$f(\theta).f(\phi)=\cos(\theta+\phi)+i\sin(\theta+\phi)$$
$$=f(\theta+\phi);$$
$$\therefore\ f(\theta)=z^\theta;$$
$$\therefore\ \{f(\theta)\}^n=\{z^\theta\}^n$$
$$=z^{n\theta}$$
$$=f(n\theta);$$
$$\therefore\ \{\cos\theta+i\sin\theta\}^n=\cos n\theta+i\sin n\theta \text{ (with the usual restrictions).}$$

Bradfield College. G. P. BLAKE.

268. [V. 2. 10.] *Historical Note* (220, 221, 222).

To draw from any two given points A, B two straight lines to a point R on a given circle, having centre at C, so that they may be equally inclined to CR.

This problem, obviously intimately connected with Optics, is found in the treatise on Optics by Alhazen, circ. 1100 A.D. It was solved geometrically by Alhazen, Vitellio and Barovius, and algebraically by Huygens, Slusius and Catalan. The solution given in Milne and Davis' *Conics*, p. 155, is due to R. Simson, who discovered it prior to 1729. The similar one published by Robins in 1739 Simson seemed to think was inspired by his own, which depends on the construction of a rectangular hyperbola given two conjugate diameters.

In Prof. Hudson's second solution, line 4, we are told to fit, mechanically, a line of given length between two given lines so that its direction shall pass through a given point. This, which belongs to the directive (*νευσις*) class of problems and is mentioned by Pappus, is dealt with by Newton in his *Universal Arithmetic* in the chapter on the linear construction of equations, where he shews that the line can be constructed mechanically either by the use of the conchoid or by means of a hyperbola and a circle, and he employs it in the graphic solution of cubic equations, and also in the trisection of an angle and the finding two mean proportionals between two given quantities. The whole chapter is very interesting, as Newton there discusses the question as to what lines besides the straight line and circle should be admitted in the solution of geometrical problems. If a mechanical solution is allowed, I would suggest the following for Alhazen's Problem :

Describe in the ordinary way, by means of two pins, string and pencil, a conic having A and B as foci, and adjust the length of the string so as to make the conic touch the circle in R (suppose). Then AR and BR are equally inclined to the common tangent at R, and $\therefore$ also to the radius CR.

JOHN J. MILNE.

269. [V. 4. b. β.] *On Note* 235.

Line 6 from bottom ; after 'most,' insert "at which the values of r and θ are the same for both curves." W. H. H. HUDSON.

270. [V. 4. b. β.] *On the intersections of the Limaçon and Quatrefoil used in the graphical solution of a biquadratic.* (*Math. Gazette*, Jan. 1907, p. 22.)

Although, using polar coordinates, there are only 4 pairs of values of r, θ that satisfy both equations, it is easy to draw a figure shewing 9 points of

intersection of the curves. Moreover, in rectangular coordinates, the equation of the Limaçon is of degree 4, and that of the Quatrefoil of degree 6; therefore there should be in all 24 points of intersection.

This apparent discrepancy is to be accounted for.

Eliminating y from the rectangular equations of the two curves, an equation may be obtained of degree 16 in x, which contains x^8 as a factor.

Hence there are 8 points at ∞, 8 at the origin and 8 others.

Of these last, 4 are determined by r, θ in one curve, and $-r$, $\pi+\theta$ in the other; therefore they do not appear in the treatment by polar coordinates.

The 8 points at the origin, arising from the intersections of the 2 branches of the Limaçon with the 4 branches of the Quatrefoil, shew as 1 in the figure; the values of θ that make $r=0$ are not the same in both curves, so that these points also do not appear in the treatment by polar coordinates.

The 8 points at ∞ are the imaginary circular points, each counted 4 times.

In order to exclude the 4 superfluous points which, besides the origin, appear in the figure and are not the representation of the solution of the biquadratic, it is well to mark with dots, or in some other way, the portions of the curves for which a negative value of r corresponds to a positive value of θ; the points where a dotted portion intersects one not dotted are excluded.

W. H. H. Hudson.

271. [K. 2. c.] *The polar circle of a triangle is cut orthogonally by the director circle of any inscribed conic.*

Let TP, TP' be tangents to a conic whose foci are S, S', centre O, axes 2α, 2β. On SP cut off $SZ=2a$. Then we know that $TZ=TS'$, $STZ=P'TP$.

Hence
$$\begin{aligned} ST\,.\,S'T\cos PTP' &= \tfrac{1}{2}(ST^2+S'T^2-4a^2) \\ &= OT^2+OS^2-2a^2=OT^2-\alpha^2-\beta^2 \\ &= \text{sq. of tangent from } T \text{ to the director circle.} \end{aligned}$$

Also
$$\begin{aligned} \tfrac{1}{2}ST\,.\,S'T\sin PTP' &= \triangle STZ \\ &= STP+S'TP \\ &= 2OTP \\ &= \text{area of quadrilateral } OP'TP. \end{aligned}$$

Thus if ABC is a circumscribed triangle, K its orthocentre and ρ the radius of its polar circle,

$$\Sigma SA\,.\,S'A\sin A = 2\triangle ABC = 4R^2\sin A\sin B\sin C,$$

and since K is the mean centre of A, B, C for multipliers $\cot C\cot B$, $\cot C\cot A$, $\cot A\cot B$, whose sum is 1,

Sq. of tangent from K to director circle
$$\begin{aligned} &= \Sigma SA\,.\,S'A\cos A\,.\,\cot B\cot C + \text{sq. of tangent from } K \text{ to circumcircle} \\ &= \cot A\cot B\cot C\,.\,\Sigma SA\,.\,S'A\sin A + 2\rho^2 \\ &= 4R^2\cos A\cos B\cos C + 2\rho^2 = \rho^2. \end{aligned}$$

Q.E.D.

A. C. Dixon.

272. [K. 2. c.] *Geometrical proof that the polar circle of a triangle is orthogonal to the director circle of the inscribed circle.*

Produce AX to T so that $AT=FE$; then since $AH=2OD$ therefore $HT=2DE$.

Since AET is a right angle

$$AT^2-IT^2=AE^2-IE^2=AE^2-CE^2=FE(NE-DE)=FE\,.\,AX;$$

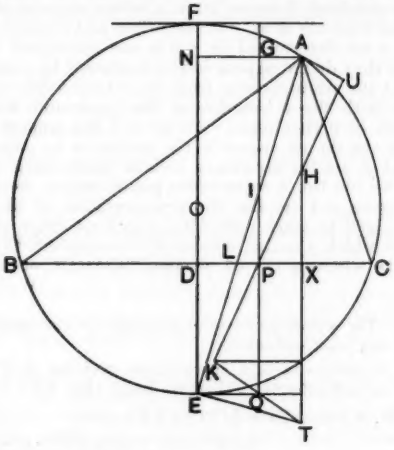
therefore
$$IT^2=AT(AT-AX)=AT\,.\,XT.$$

Again, completing the rectangle $EDPQ$, since $IE^2=AE\,.\,LE$; therefore

$$IQ^2=GQ\,.\,PQ.$$

Thus T and Q are points on the radical axis of a point-circle at I and the circle upon AP as diameter.

Now $HT=2DE=2PQ$; consequently if HP be produced to K so that $HP=PK$ then K is also a point on the radical axis TQ.



Drawing AU perpendicular to HPK, $IK^2=PK.UK$.

But $$IH^2+IK^2=2IP^2+2HP^2;$$

therefore
$$\begin{aligned} 2IP^2-IH^2&=PK.UK-2HP^2\\ &=HP(UH+2HP)-2HP^2\\ &=HP.UH\\ &=AH.HX. \end{aligned}$$

R. F. Davis.

273. [K. 1. c.] *Note on Euclid I.* 32. *Corollary.*

If one vertex of a polygon of n sides be joined to the others the number of triangles formed is $n-2$.

Each triangle contains angles whose sum is 2 right angles.

$\therefore$ the sum of all the interior angles of the polygon $=2(n-2)$ right angles.

A. A. Bourne.

274. [A. 1. b.] *Note on the Power Inequality.* [Cf. Mr. V. R. Aiyar, p. 322.]

In proving the fundamental inequality that if $p>q$ and $x>y$, then

$$\frac{p}{q}x^{p-q}>\frac{x^p-y^p}{x^q-y^q}>\frac{p}{q}y^{p-q}$$

for the simple case when p and q are integers, it is well to notice that the statement can be written

$$\frac{x^{p-q}(x^q-y^q)}{(x-y)q}>\frac{x^p-y^p}{(x-y)p}>\frac{y^{p-q}(x^q-y^q)}{q(x-y)},$$

which is equivalent to the following truism:

In a Geometrical Progression $x^{p-1}+x^{p-2}+\ldots y^{p-1}$ the average of all the terms is intermediate between the average of the first q and the average of the last q.

F. J. W. Whipple.

275. [L. 4. a.] *A new method of finding the condition that the general conicoid should be one of revolution and of finding the equation of its axis.*

In Note 200 of the *Mathematical Gazette* (May 1906) I gave a new method of finding the condition that the general conicoid should be one of revolution and of finding the equation of its axis. The following method is shorter and certain exceptional cases are treated.

Let the conicoid be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0. \quad \ldots\ldots(2)$$

Then $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d$
$$+(lx+my+nz+p)^2=0 \quad \ldots\ldots(2)$$

is the system of conicoids having contact with (1) round the curves of section by the planes
$$lx+my+nz+p=0. \quad \ldots\ldots(3)$$

If the conicoid (1) be one of revolution and the plane of section be perpendicular to its axis, in which case l, m, n will be the proportional to the direction cosines of that axis, one of the conicoids having contact with (1) round the curve of section by that plane will be a sphere.

The conditions that (2) should be a sphere are

$$a+l^2=b+m^2=c+n^2, \quad f+mn=0, \quad g+nl=0, \quad h+lm=0.$$

Thus if f, g, h are all non-zero

$$l^2=-\frac{gh}{f}, \quad m^2=-\frac{hf}{g}, \quad n^2=-\frac{fg}{h},$$

and the required condition is

$$a-\frac{gh}{f}=b-\frac{hf}{g}=c-\frac{fg}{h};$$

but if one of them, say g, is zero, then $l=0$ or $n=0$, say $l=0$, then $h=0$, and we have the condition $(a-b)(a-c)=f^2$.

Finally if all three are zero, two of the quantities l, m, n must be zero, *i.e.* two of the quantities a, b, c must be equal, which is the required condition in this case.

Again the equation of the axis of revolution is

$$\frac{x-\frac{U}{D}}{l}=\frac{y-\frac{V}{D}}{m}=\frac{z-\frac{W}{D}}{n}.$$

But from the above work we have the following:—If f, g, h are all non-zero,

$$l:m:n=1/f:1/g:1/h;$$

if two of them, g and h, are zero, $l=0$, $m=\sqrt{a-b}$, $n=\sqrt{a-c}$ (the signs before the roots being chosen so as to make $mn+f=0$); and if all three are zero, in the case where the two equal coefficients among a, b, c are b and c we have $l=\sqrt{b-a}$, $m=0$, $n=0$. Thus the equation of the axis of revolution is

$$\frac{x-\frac{U}{D}}{1/f}=\frac{y-\frac{V}{D}}{1/g}=\frac{z-\frac{W}{D}}{1/h},$$

where f, g, h are all non-zero.

$$\frac{x-\frac{U}{D}}{0}=\frac{y-\frac{V}{D}}{\sqrt{a-b}}=\frac{z-\frac{W}{D}}{\sqrt{a-c}},$$

where $g=0$ and $h=0$, the sign given to the radicals being such as to make $\sqrt{a-b}.\sqrt{a-c}$ have the opposite sign to f;

and
$$\frac{x-\frac{U}{D}}{1}=\frac{y-\frac{V}{D}}{0}=\frac{z-\frac{W}{D}}{0},$$

where $f=0$, $g=0$, $h=0$ and the two equal quantities among a, b, c are b and c.

H. L. TRACHTENBERG.

ANSWER TO QUERY.

[52, p. 295.] General solution of $x^2+y=a$, ...i.
$y^2+x=b$. .....ii.

Let
$$c^6-4ac^4+4bc^2-1=0;$$

whence c can be found in terms of a and b.

Substituting from i. in ii.,
$$4c^2(a-x^2)^2+4c^2x=4c^2b=1+4ac^4-c^8;$$

whence
$$2c\left\{x^2-\left(a-\frac{c^2}{2}\right)\right\}=\pm(2c^2x-1).$$

W. H. LAVERTY.

BOOKS, ETC., RECEIVED.

American Journal of Mathematics. Edited by F. MORLEY. Vol. XXX. No. 2. April 1908. $5 per ann. (4 parts a year).

Concerning Systems of Conics lying on Cubic, Quartic, and Quintic Surfaces. C. H. SISAM. *On the Canonical forms and Automorphs of Ternary Cubic Forms.* L. E. DICKSON. *The Elliptic Cylinder Function of Class K.* W. H. BUTTS. *On Elliptic Modular Equations for Transformations of Orders 29, 31, 37.* ARTHUR BERRY. *On Translation-Surfaces connected with Unicursal Quartic.* J. EIESLAND.

Bulletin of the Mathematical Society. Edited by F. MORLEY and others. Vol. XIV. No. 7. April 1908. $5 a year. (Macmillan Co.)

Subjective Geometry. G. W. HILL, *On Higher Congruences and Molecular Invariants.* L. E. DICKSON. *Note on Jacobi's Equation in the Calculus of Variations.* M. MASON. *On the Distance from a Point to a Surface.* E. R. HEDRICK. *A Geometric Representation of the Galois Field.* L. I. NEIKIRK. *The Degree of an Irreducible Linear Homogeneous Group.* W. B. FITE. *On the Lorentzian Transformation and the Radiation from a Moving Electron.* F. R. SHARPE.

Die Nichteuklidische Geometrie. By R. Bonola. Translated into German by H. Liebmann. Pp. viii, 244. 5 m. 1908. (Teubner.)

Vorlesungen über Geschichte der Mathematik. Vol. IV. (1759-1799.) Edited by M. Cantor. Pp. 1113. 35 m. 1908. (Teubner.)

Historical Survey. S. GÜNTHER. *Arithmetic, Algebra, Theory of Numbers.* F. CAJORI. *Combinations, Probabilities. Imaginary Series.* E. NETTO. *Elementary Geometry.* V. BOBYNIN. *Trigonometry, Polygonometry.* A. v. BRAUNMÜHL. *Analytical Plane and Solid Geometry.* V. KOMMERELL. *Perspective and Descriptive Geometry.* G. LORIA. *Infinitesimal Calculus.* G. VIVANTI. *Differential Equations. Calculus of Differences. Calculus of Variations.* C. R. WALLNER. *General Synopsis.* 1758-1799. M. CANTOR.

Annals of Mathematics. 2nd Series. Vol. IX. No. 3. April 1908. $2 a vol. (Longmans.)

A Necessary Condition that all Roots of an Equation are Real. O. D. KELLOGG. *The Equilibrium of a Heavy Homogeneous Chain in a Uniformly Rotating Plane.* E. B. WILSON. *The Continuity of the Roots of an Algebraic Equation.* J. L. COOLIDGE. *On the Differentiation of Finite Integrals.* W. F. OSGOOD. *Note on the Convergence of a Sequence of Functions of a Certain Type.* H. E. BUCHANAN and T. H. HILDEBRANDT. *On the Inverse Problem of a Calculus of Variations.* G. A. BLISS. *On the Geometrical Interpretation of the Generalized Law of the Mean.* C. N. HASKINS. *On the Torsion of a Curve.* P. SAUREL.

Bedingungsgleichungen für Liniennetze und für Rückwärtseinschnitte. L. Krüger. Pp. 50. 4 m. 1908. (Teubner.)

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